HOMOTOPY PROPERTIES OF THE SPACE OF HOMEOMORPHISMS ON P^2 AND THE KLEIN BOTTLE(1)

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1. **Introduction.** This is another in a series of papers dealing with the computation of the homotopy groups of the space of homeomorphisms on a 2-manifold, M, (the homeotopy groups of McCarty [12]). Denote by $H(M,K)$ the space of homeomorphisms on M leaving K pointwise fixed and by $H_0(M,K)$ its identity component. Let $\dot{M}$ denote the boundary of the manifold M. Then if M is a disc with holes, $H_0(M,\dot{M})$ is homotopically trivial ([1] and [4]) and if M is a torus the homotopy groups of $H_0(M)$ are isomorphic to those of M [6]. Quintas [14] related $H(M)$ to $H(M,x)$, where M is a 2-manifold and x is a point of M and McCarty [11] related $H(M,x)$ to $H(M-x)$. He also introduced some fiber space techniques to the study of these problems. Lickorish studied some of these problems also in his recent interesting papers [9] and [10]. Local homotopy properties are considered by McCarty in [11] and Dyer and myself in [3]. In the present paper the Klein bottle, Moebius strip and P^2 are considered. In a paper under preparation I prove that $H_0(M)$ has trivial homotopy groups if M is orientable and has two or more handles, or is nonorientable and has three or more cross caps.

2. **The Moebius strip.** Denote the Moebius strip by M in the rest of this paper. The space $H(M)$ has two components. To see that there are at least two, let M be obtained from $I \times I$ by identifying appropriate points of $I \times 0$ and $I \times 1$ and let a homeomorphism f of M be obtained by reflecting $I \times I$ on $I \times 1/2$ before the identification is made. Then f reverses orientation on $\dot{M}$ and is thus not isotopic to the identity. The Moebius strip may also be obtained from $I \times S^1$ by identifying diametrically opposite points of $0 \times S^1$ and f may be obtained by first reflecting $I \times S^1$ on a diameter and then making the identification.

THEOREM 2.1. *The space $H(M,\dot{M})$ is homotopically trivial.*

Proof. First, $H(M,\dot{M})$ is arcwise connected. To see this, let I_x denote the image of $I \times x$ under the first identification map described above and suppose

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that g is an element of $H(M,\dot{M})$. An isotopy leaving the endpoints fixed can easily be constructed that carries $g(I_0)$ back to an arc α that is either I_0 or has the properties that it intersects I_0 only a finite number of times, it crosses I_0 at each point of intersection except the endpoints, and at the endpoints of the closure k of each component of $\alpha - I_0$, k abuts on I_0 from opposite sides. It is fairly easily seen that this second case cannot occur so that there is an isotopy h_t of I_0 in M such that $h_0 = g$, $h_1 = i$ and $h_t \mid \dot{I}_0 = i$.

In $I \times M$, let G be a homeomorphism of $((0 \cup 1) \times M) \cup (I \times (\dot{M} \cup I_0))$ into $I \times M$ such that $G(t,x) = (t, h_t(x))$ for x in I_0, $G(t,x) = (t,x)$ for x in $\dot{M}$, $G(1,x) = (1,x)$ for each x and $G(0,x) = (0, g(x))$. The space of homeomorphism of M onto itself leaving $\dot{M} \cup I_0$ pointwise fixed is clearly homeomorphic to the space of homeomorphisms of a disc into itself leaving its boundary pointwise fixed. Therefore, it follows from [2] that G may be extended to a homeomorphism G^* of $I \times M$ onto itself leaving (t, M) invariant for each t. Let g_t denote the homeomorphism of M onto itself taking x onto y, where $G^*(t,x) = (t,y)$. This yields the required arc from g to i in $H(M,\dot{M})$.

Now suppose f is a mapping of S^k into $H(M,\dot{M})$. Consider $I \times E^1$ and the standard covering map π taking it onto $M, \pi(I \times n)$ being I_0, where n is an integer and, in general, $\pi(I \times x')$ being I_x, where $x' \equiv x \bmod 1$. Let P_0 and P_1 denote the endpoints of I_0 and Q_0 and Q_1 the points of $I \times 0$ going into P_0 and P_1 under π. Let F be the mapping of $I_0 \times S^k$ into M defined by $F(x,y) = f(y)(x)$. Then $F(P_0, y) = P_0$ and $F(P_1, y) = P_1$ for each point y of S^k. The space $P_0 \times S_k$ is a strong deformation retract of $I_0 \times S^k$. Standard lifting theorems [7, p. 63] yield a map G of $I_0 \times S^k$ into $I \times E^1$ such that $G(P_0 \times S^k) = Q_0$ and $\pi G(x,y) = F(x,y)$ for each (x,y). Then $G \mid (I_0 \times y)$ is the unique lifting of $F \mid I_0 \times y$ to a map of $I_0 \times y$ into $I \times E^1$ taking (P_0, y) onto Q_0. This is a homeomorphism. Also, since $F \mid (I_0 \times y)$ is isotopic to a mapping taking (x,y) onto x under an isotopy leaving $F((P_0 \cup P_1), y)$ unchanged, lifting theorems yield the fact that $G(P_1 \times S^k) = Q_1$.

A proof almost identical to that of a similar theorem for tori [6] now yields a mapping z of $I_0 \times S^k \times I$ into M such that $z(x,y,0) = F(x,y)$ and $z(x,y,1) = x$. Let α be the homeomorphism of $[M \times S^k \times (0 \cup 1)] \cup [(I_0 \cup \dot{M}) \times S^k \times I]$ into $M \times S^k \times I$ such that $\alpha(x,y,t) = (x,y,t)$ for $x \in \dot{M}$, $\alpha(x,y,t) = (z(x,y,t), y, t)$ for $x \in I_0$, $\alpha(x,y,0) = (f(y)(x), y, 0)$, and $\alpha(x,y,1) = (x,y,1)$. Since the space of homeomorphisms of the disc onto itself leaving its boundary pointwise fixed is contractible, it follows from [2] that α can be extended to a homeomorphism α^* of $M \times S^k \times I$ onto itself that leaves each (M, y, t) invariant. Then if Z_t is the map of S^k into $H(M,\dot{M})$ such that $Z_t(y)(x) = x'$, where $\alpha^*(x,y,t) = (x', y, t)$, Z_t is the required homotopy taking f into a constant mapping.

THEOREM 2.2. *For* $k > 1$, $\pi_k(H_0(M)) = 0$, $\pi_1(H_0(M)) = Z$, *and* $H(M)$ *has exactly two components.*

Proof. Let p denote the map of $H(M)$ into $H(\dot{M})$ such that $p(f) = f | \dot{M}$ for each $f \in H(M)$. Then $H(M)$ is a fiber space with projection p, base space $H(\dot{M})$ and fiber $H(M, \dot{M})$ (in the sense of Hu [7, p. 62]). To see this, let g be a mapping of the k-cell R^k, $k \geqq 0$, into $H(M)$ and G_t a homotopy in $H(\dot{M})$ such that $G_0(x) = pg(x)$ for each x. Let α be a homeomorphism of $(\dot{M} \times R^k \times I) \cup (M \times R^k \times 0)$ into $(M \times R^k \times I)$ such that $\alpha(y, x, t) = (G_t(x)(y), x, t)$ for y in $\dot{M}$ and $\alpha(y, x, 0) = (g(x)(y), x, 0)$. As in the proof of Theorem 2.1 α may be extended to a homeomorphism α^* of $M \times R^k \times I$ onto itself and a homotopy g_t on $H(M)$ is constructed such that for each t, $g_t(x) | \dot{M} = G_t(x) | \dot{M}$ for each x in R^k. Thus $pg_t(x) = G_t(x)$ and $H(M)$ is a fiber space. The homotopy sequence of this fibering now yields the first part of the theorem (Hu [7, p. 152]), since $\pi_i H(M, \dot{M}) = 0$ for $i \geqq 1$.

If f is a map in $H(M)$, then $p(f)$ lies in one of the two homeomorphic components of $H(\dot{M})$. If $p(f)$ and $p(f')$ lie in the same component of $H(\dot{M})$, there is an arc from $p(f)$ to $p(f')$ in that component, and thus there is an arc in $H(M)$ from f to a map f'' such that $p(f'') = p(f')$. Since $H_0(M, \dot{M})$ is arcwise connected, there is an arc from f to f' in $H(M)$. This proves that there are at most two components of $H(M)$. That there are at least two components was proved in the opening paragraph of this section.

3. The projective plane.

THEOREM 3.1. *The space $H(P^2)$ has only one component, $H(P^2, p)$ has two components, $\pi_i H(P^2, p) = 0$ for $i > 1$, and $\pi_1 H(P^2, p) = Z$.*

Proof. That $H(P^2)$ has only one component was observed by Lickorish in [10], but it follows readily from the following argument. Let D be a disc (consider it circular) in P^2 with center p and M the Moebius strip in P^2 with boundary $\dot{D}$. Let f be an element of $H(P^2, p)$ that takes $\dot{D}$ onto itself, but reverses orientation. If f were isotopic to i under an isotopy leaving p fixed, $f | \dot{D}$ would be homotopic to the identity in $P^2 - p$, which is impossible. Thus $H(P^2, p)$ has at least two components.

Let f be an element of $H(P^2, p)$, E a disc with center p lying in $(\operatorname{int} D) \cap (\operatorname{int} f(D))$ and g a homeomorphism in $H(P^2, p)$ such that $g | \dot{D} = f | \dot{D}$ and $g | E$ is either the identity or a reflection on a diameter. The map $g^{-1} f = i$ on $\dot{D}$. Thus there is an isotopy F_t such that $F_t | \dot{D} \cup p = i$, $F_1 = g^{-1} f$ and $F_0 = i$. The map $g | \dot{E}$ is either the identity or a reflection. There is thus an isotopy G_t such that $G_t | E = g | E$, $G_1 = g$ and G_0 is either the identity or is, on $P^2 - E$, the reflection described in §2. Then $Z_t = G_t F_t$ is an isotopy such that $Z_1 = f$ and Z_0 is the identity or the "reflection" described in the foregoing parapgraph. Thus $H(P^2, p)$ has two components. This reflection is isotopic to the identity in $H(P^2)$, as can be seen by pushing p around an orientation reversing simple closed curve so that the reflection is isotopic to a map f that leaves p fixed, but gives rise to a map g that is the identity on E. The above argument was suggested by Roberts [15].

Let f be a map of S^k into $H(P^2,p)$ and E a circular disk with center p lying in int D and in int$f(x)(D)$ for each x. It follows from Theorem 2.9 of [5] and the techniques of §2 that there is a map g of S^k into $H(P^2,p)$ such that $g(x)\,|\,M = f(x)\,|\,M$ if $k > 1$, $g(x)\,|\,E = i$, and in any case $g(x)(E) = E$. The map $[g(x)]^{-1}f(x)\,|\,\dot{D} = i$, so there is a map F of $S^k \times I$ into $H(P^2,p)$ such that $F(x,1) = [g(x)]^{-1}f(x)$ and $F(x,0) = i$. Also, if $k > 1$, $g(x)\,|\,E = i$, so there is a mapping Z of $S^k \times I$ into $H(P^2,p)$ such that $G(x,1) = g$ and $G(x,0) = i$. If $k = 1$, let $G(x,t) = g(x)$ for each x and t. Then, $Z(x,t) = G(x,t)F(x,t)$ yields a map of $S^k \times I$ into $H(P^2,p)$ such that $Z(x,1) = f(x)$ and $Z(x,0) = i$ if $k > 1$ and $g(x)$ if $k = 1$. The proof of the theorem for $k > 1$ is complete.

The map g_1 of S^1 into $H(E)$ such that $g_1(x) = g(x)\,|\,E$ represents an element of $\pi_1 H(E)$. If h is a map of S^1 into $H(P^2,p)$ such that h_1 and g_1 represent different elements of $\pi_1 H(E)$, then the maps h_2 and g_2 of S^1 into $H(\dot{E})$ such that $h_2(x) = h_1(x)\,|\,\dot{E}$ and $g_2(x) = g_1(x)\,|\,\dot{E}$ represent different elements of $\pi_1 H(\dot{E})$. If q is a point of $\dot{E}$, the maps h_3 and g_3 of S^1 into $\dot{E}$ such that $h_3(x) = h_2(x)(q)$ and $g_3(x) = g_3(x)(q)$ represent different elements of $\pi_1(\dot{E})$ (see [12, §6]). However, if h and g represent the same element of $\pi_1 H(P^2,p)$, h_3 and g_3 are homotopic in $P^2 - p$, which is impossible. This demonstrates that $\pi_1 H(P^2,p) = \pi_1(\dot{E}) = Z$ and completes the proof of Theorem 3.1.

THEOREM 3.2. *For* $i > 2$, $\pi_i H(P^2) = \pi_i(P^2)$, $\pi_2 H(P^2) = 0$ *and* $\pi_1 H(P^2) = Z_2$.

Proof. It follows from Lemma 4.1 of [12] that $H(P^2)$ is a fiber space with base space P^2 and fiber $H(P^2,x)$, the projection map p carrying each homeomorphism h into $h(x)$. Since $\pi_i H(P^2,x) = 0$ for $i \geqq 2$, the homotopy sequence for this fiber space yields the required isomorphism for $i > 2$. The lower end of this sequence is

$$\cdots \xrightarrow{d_3} \pi_2 H(P^2,x) \xrightarrow{i_2} \pi_2 H(P^2) \xrightarrow{p_2} \pi_2(P^2,x) \xrightarrow{d_2} \pi_1 H(P^2,x)$$
$$\xrightarrow{i_1} \pi_1 H(P^2) \xrightarrow{p_1} \pi_1(P^2,x) \xrightarrow{d_1} \pi_0 H(P^2,x) \xrightarrow{i_0} \pi_0 H(P^2).$$

Since $H(P^2)$ is connected, d_1 is onto and is 1-1, since $\pi_1(P^2,x) = Z_2$. Also d_1 has kernel 0. Therefore i_1 is onto. A closer look at P^2 is required for further information.

Let x' and y' be diametrically opposite points of S^2 and ϕ the standard covering map of S^2 into P^2 taking x' onto x. Let $H^0(S^2)$ be the space of rigid motions of S^2 and $H^0(S^2,x')$ those leaving x' fixed. Then $H^0(S^2)$ is a fiber space with base space S^2 and fiber $H^0(S^2,x')$. Let ϕ^* be the map of $H^0(S^2)$ into $H(P^2)$ that takes each element f' of $H^0(S^2)$ into the homeomorphism f taking each point z of P^2 onto $\phi f' \phi^{-1}(z)$. This makes sense since $\phi^{-1}(z)$ is a pair of diametrically opposite points, as is $f'\phi^{-1}(z)$. Note that ϕ^* takes fibers into fibers and $p\phi^* = \phi p'$. For each i, ϕ^* and ϕ induce homomorphisms of $\pi_i H^0(S^2)$, $\pi_i H^0(S^2, x')$, $\pi_i(S^2)$ onto $\pi_i H(P^2)$, $\pi_i H(P^2, x)$, $\pi_i(P^2)$, and in particular, the following diagram

$$\begin{array}{ccccccccc}
\cdots \to & \pi_2 H^0(S^2) & \xrightarrow{p'_2} & \pi_2(S^2, x') & \xrightarrow{d'_2} & \pi_1 H^0(S^2, x') & \xrightarrow{i'_1} & \pi_1 H^0(S^2) & \xrightarrow{p'_1} \pi_1(S^2, x') \to \cdots \\
& \downarrow \phi_2^* & & \downarrow \phi_2 & & \downarrow \phi_1^{**} & & \downarrow \phi_1^* & \\
\cdots \to \pi_2 H(P^2, x) \xrightarrow{i_2} & \pi_2 H(P^2) & \xrightarrow{p_2} & \pi_2(P^2, x) & \xrightarrow{d_2} & \pi_1 H(P^2, x) & \xrightarrow{i_2} & \pi_1 H(P^2) & \to \cdots
\end{array}$$

where ϕ_2^*, ϕ_2, etc., are the induced homomorphisms, is commutative. It is observed in [3] and [12] that $\pi_1 H^0(S^2) = Z_2$ and $\pi_2 H^0(S^2) = 0$. Also, ϕ_2 is an isomorphism, $\pi_1 H^0(S^2, x') = Z$, and the diagram looks as follows

$$\begin{array}{ccccccccc}
\cdots \to & 0 & \xrightarrow{p'_2} & Z & \xrightarrow{d'_2} & Z & \xrightarrow{i'_1} & Z_2 & \xrightarrow{p'_1} 0 \to \cdots \\
& \downarrow \phi_2^* & & \downarrow \phi_2 & & \downarrow \phi_1^{**} & & \downarrow \phi_1^* & \\
\cdots \to 0 \xrightarrow{i_2} & \pi_2 H(P^2) & \xrightarrow{p_2} & Z & \xrightarrow{d_2} & Z & \xrightarrow{i_1} & \pi_1 H(P^2) & \to \cdots .
\end{array}$$

The image of Z under i'_1 is Z_2. Hence the kernel of i'_1 is 2Z. Then d'_2 is a 1-1 map of Z onto 2Z. Since ϕ_2 is an isomorphism and $d_2\phi_2 = \phi_1^{**} d'_2$, $d_2(Z) = 2Z$. (It is an easy exercise to prove that ϕ_1^{**} is also an isomorphism.) Therefore the kernel of i_1 is 2Z and, since i_1 is onto, $\pi_1 H(P^2) = Z_2$. Since d_2 is 1-1, i_2 is onto. Hence $\pi_2 H(P^2) = 0$.

REMARK. The facts that ϕ_i^{**} is an isomorphism for $i \geqq 1$ and ϕ_i is an isomorphism for $i \geqq 2$ yield, by the five lemma, ϕ_i^* is an isomorphism of $\pi_i H^0(S^2)$ onto $\pi_i H(P^2)$ for $i \geqq 1$. But $H^0(S^2)$ is homeomorphic to P^3. Therefore $\pi_i H(P^2) = \pi_i(P^3)$ for all i. Also $\pi_i H(P^2) = \pi_i(S^3)$ for $i > 1$, and for $i > 2$, $\pi_i H(P^2) = \pi_i(S^2) = \pi_i(P^2)$. (See the opening remarks of [3].)

4. **The Klein bottle.** The Klein bottle has two representations, A and B. Let A be the representation obtained from the annulus $S^1 \times I$ by identifying diametrically opposite points of $S^1 \times 0$ and $S^1 \times 1$. Let C be the curve $S^1 \times 1/2$. Then K is the union of two Moebius strips M_0 and M_1, sewn together along their common boundary, C. Their center curves, C_0 and C_1 are obtained from the identification of points of $S^1 \times 0$ and $S^1 \times 1$. Coordinatize S^1 by the reals mod 1 and let C' be the closed curve in K obtained from $(0 \times I) \cup (1/2 \times I)$ by means of the identification. Lickorish [10] has shown that $\pm C, C'$, and a curve bounding a disc represent the four isotopy classes of orientation preserving simple closed curves on K. The four classes of orientation reversing curves are represented by $\pm C_0$ and $\pm C_1$.

Let B be the representation obtained from $S^1 \times I$ by identifying $(x,0)$ and

$(-x,1)$. In this representation, C' is represented by $S^1 \times 1/2$, C by $(1/4 \times I) \cup (3/4 \times I)$, C_0 by $0 \times I$ and C_1 by $1/2 \times I$.

The fundamental group, $\pi_1(K)$ is generated by the classes of the homeomorphisms α_0 and α_1 of S^1 into C_0 and C_1 (assume base point to be on C_0) and $\pi_1(K)$ is isomorphic to the free group generated by $[\alpha_0]$ and $[\alpha_1]$ with the sole relation $[\alpha_0]^2[\alpha_1]^2 = 1$. The group $\Lambda(K)$, the quotient group of $H(K) \bmod H_0(K)$ is, as Lickorish points out, isomorphic to $Z_2 \times Z_2$. Thus $H(K)$ has four components. These (see [10]) are generated by the identity homeomorphism, i, the homeomorphism f obtained from representation B by reflecting $S^1 \times I$ on $S^1 \times 1/2$ and then identifying, the homeomorphism h obtained by cutting K along C', twisting through 360° and sewing back together, and hf.

THEOREM 4.1. *The space $H(K,x)$ $(x \in C')$ has infinitely many components, $H(K)$ has four components, $\pi_i H(K,p) = 0$ for each $i \geqq 1$, $\pi_i H(K) = 0$ for $i \geqq 2$ and $\pi_1 H(K) = Z$.*

Proof. A covering space for K is $S^1 \times E^1$. The covering map π is such that $\pi(S^1,t) = \pi(S^1,t')$ if $t \equiv t' \bmod 1$. The proof that $H(K,x)$ has trivial homotopy groups is the same as the proof in [6] that $H(T^2,p)$ has trivial homotopy groups. Let $H^0(K,x)$ denote the subset of $H(K,x)$ consisting of the elements that are in $H_0(K)$. Then g is in $H^0(K,x)$, where g is the homeomorphism obtained either from representation B by first reflecting on the diameter containing $(0 \cup 1/2) \times I$ and then identifying or from representation A by first rotating through π radians and then identifying. From the last form of g, it can be easily seen that g is in $H_0(K)$. Let $Z(t)$ be the homeomorphism of K obtained by rotating $S^1 \times I$ through πt radians and then identifying. Then $Z(1) = g$ and $Z(0) = i$. Also, k is in $H^0(K,x)$, where k is obtained from representation B by pushing x clockwise around C' through 360° while leaving $(S^1 \times 0) \cup (S' \times 1)$ pointwise fixed and then identifying. The group structure of $H^0(K,x)$ appears below.

It again follows from [12] that $H(K)$ is a fiber space with fiber $H(K,x)$ and base space K. The lower part of the exact homotopy sequence for this fiber space is

$$\cdots \to \pi_1 H(K,x) \overset{i_1}{\to} \pi_1 H(K) \overset{p_1}{\to} \pi_1(K) \overset{d_1}{\to} \pi_0 H(K,x) \overset{i_0}{\to} \pi_0 H(K).$$

Since $\pi_i(K) = 0$ for $i \geqq 2$, it follows that $\pi_i H(K) = \pi_i H(K,x) = 0$ for $i \geqq 2$. Since $\pi_1 H(K,x) = 0$, p_1 is 1-1. The kernel of i_0 is $H^0(K,x)$, so this is the image of d_1. Since $\pi_1 H(K)$ is isomorphic to the kernel of d_1, knowledge of this kernel yields the structure of $\pi_1 H(K)$. A study of the meaning of d_1 shows that, with the base point for $\pi_1(K)$ on C_0, $d_1[\alpha_0] = [g]$, the isotopy class in $H(K,x)$ of g, $d_1[\alpha_1] = [kg]$, but $d_1[\alpha_0]^2 = d_1[\alpha_1]^2 = [i]$. Since $[gkg] = [k^{-1}]$, it follows readily that the infinite cyclic subgroup of $\pi_1(K)$ generated by $[\alpha_0]^2$ $(= [-\alpha_1]^2)$ is the kernel of d_1. Thus $\pi_1 H(K) = Z$ and it follows that $\pi_0 H^0(K,x)$ is isomorphic to the free group on two generators a, b with the relation $a^2 = b^2 = 1$.

5. **Appendix on generators and Lie groups.** If the Moebius strip M is considered as being obtained from the annulus $S^1 \times I$ by the identification of diametrically opposite points of $S^1 \times 0$, the "rotations" of M obtained by first rotating $S^1 \times I$ and then identifying form a subgroup $L_1(M)$ of $H_0(M)$ whose fundamental group is Z. The arguments in §2 indicate that the injection of $L_1(M)$ into $H_0(M)$ induces an isomorphism of the fundamental groups.

In [8], Kneser proved that the space of rotations of S^2, SO(3) (the identity component of my $H^0(S^2)$), is a deformation retract of $H_0(S^2)$. This implies the results mentioned in the remark at the end of §3. The mapping ϕ^* is a homeomorphism of the identity component of the space of antipodal homeomorphisms of S^2 onto $H(P^2)$ and $\phi^* \mid \mathrm{SO}(3)$ takes SO(3) onto a group, $L(P^2)$. Since ϕ_n^* is an isomorphism of $\pi_n H^0(S^2)$ onto $\pi_n H(P^2)$, it follows that the injection map of $L(P^2)$ into $H(P^2)$ induces isomorphisms of the homotopy groups.

It is also true that $H^0(S^2,x')$ is a deformation retract of $H(S^2,x')$. The identity component, $H_0^0(S^2,x')$ is simply the group of rotations about the diameter through x'. Let $L(P^2,x) = \phi^* H_0^0(S^2,x')$. Clearly the elements of $L(P^2,x)$ are obtained by rotating D and M (see first paragraph of §3). The injection of $L(P^2,x)$ into $H_0(P^2,x)$ clearly induces isomorphisms of the homotopy groups.

Finally, let $L_1(K)$ be the subgroup of $H(K)$ consisting of the "rotations" obtained by first rotating $S^2 \times I$ in representation B and then identifying. The fundamental group of $L_1(K)$ is Z and it should be clear from §4 that the injection into $H(K)$ induces isomorphisms of the homotopy groups.

The group SO(3) is a transitive Lie group operating on S^2. Also, $L(P^2)$ is a transitive Lie group operating on P^2, so it is of interest that the injections induce isomorphisms of the homotopy groups. It is seen in [6] that if $T = S^1 \times S^1$, then the subgroup $L(T)$ generated by the rotations of the two copies of S^1 has the same homotopy groups as $H(T)$. It is a transitive Lie group acting on the torus and the injection of $L(T)$ into $H(T)$ induces isomorphisms of the homotopy groups.

Mostow proves in [13] that the only other compact 2-manifold on which a Lie group acts transitively is the Klein bottle, K. This Lie group, $L(K)$ has Z as its fundamental group and the higher homotopy groups are trivial. Also, $L(K)$ has $L_1(K)$ as a subgroup. The injection of $L_1(K)$ in $L(K)$ induces an isomorphism of the fundamental groups. It would seem, considering the fact that SO(3) is a deformation retract of $H_0(S^2)$, that $L(P^2)$, $L(T)$ and $L(K)$ are deformation retracts of $H_0(P^2)$, $H_0(T)$ and $H_0(K)$. I leave this as a conjecture.

McCarty [11, Theorem 4.4] proves that if x is a point of a compact manifold with boundary, M, then $\pi_i H(M,x)$ is isomorphic to $\pi_i H(M - x)$, the latter group having the compact open topology. The plane, E^2, the open annulus A and the open Moebius strip M° [13] are the only noncompact 2-manifolds on which transitive Lie groups act. The group $L(M^\circ)$ is isomorphic to $L(K)$ and has the "rotation" group, $L_1(M^\circ)$ as a deformation retract. It follows from

McCarty's result and Theorem 3.1, that $H(M^\circ)$ is isomorphic to Z and the comments of the earlier part of this section imply that the injection of $L_1(M^\circ)$ into $H(M^\circ)$ (thus that of $L(M^\circ)$ into $H(M^\circ)$) induces an isomorphism of the fundamental group.

Kneser also proved that $H_0(E^2)$ has as a deformation retract the space of rotations about the origin. The simplest Lie group acting transitively on E^2 is the translation group, which is clearly homotopically trivial. However, the group of rigid motions, $L(E^2)$ also acts transitively. This has the group of rotations about the origin as a deformation retract. Thus the injection of $L(E^2)$ into $H_0(E^2)$ induces isomorphisms of the homotopy groups.

Finally, McCarty's work implies that for each i, $\pi_i H_0(A)$ is isomorphic to $\pi_i H_0(E^2, 0)$. It follows that if $L_1(A)$ denotes the group of rotations of A, the injection into $H_0(A)$ induces isomorphisms of the homotopy groups. The Lie group $L(A)$ operating transitively on A has $L_1(A)$ as a deformation retract. Thus the injection of $L(A)$ into $H_0(A)$ induces isomorphisms of the homotopy groups.

Note(2). The group $L(K)$ may be roughly described as follows. Consider E^2 as a covering space of K. Identify (x, y) with $(x, y+1)$ for each (x, y) and (x, y) with $(x+\pi, -y)$. Then the mapping of K obtained by taking each point of (x, y) into $(x+a, u, \sin(x+a) + v\cos(x+a) + y)$ is a homeomorphism. The collection of such homeomorphisms for real a, w, v is a Lie group acting transitively. on K. This is $L(K)$.

If the points (x, y) and $(x+\pi, -y)$ only are identified, M° is the identification space and the same homeomorphisms of E^2 yield $L(M^\circ)$. If (x, y) is identified with $(x+2\pi, y)$, the identification space is A and the same homeomorphisms of E^2 yield $L(A)$. In each of there cases, the rotation group is a deformation retract of the transitively acting Lie group.

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